AIP | Journal of Mathematical Physics

Escort mean values and the characterization of power-law-decaying probability densities

C. Tsallis, A. R. Plastino, and R. F. Alvarez-Estrada

Citation: Journal of Mathematical Physics **50**, 043303 (2009); doi: 10.1063/1.3104063
View online: http://dx.doi.org/10.1063/1.3104063
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/50/4?ver=pdfcov
Published by the AIP Publishing

Escort mean values and the characterization of power-law-decaying probability densities

C. Tsallis,[1,2,a)] A. R. Plastino,[3,4,6] and R. F. Alvarez-Estrada[5]
[1]*Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil*
[2]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*
[3]*Physics Department, University of Pretoria, Pretoria 002, South Africa*
[4]*National University La Plata, Casilla de Correo 727, La Plata 1900, Argentina*
[5]*Departamento de Física Teórica I, Facultad de Ciencias, Universidad Complutense de Madrid, 28040, Madrid, Spain*
[6]*Instituto Carlos I de Fisica Teorica y Computacional, Universidad de Granada, 18071, Granada, Spain*

(Received 20 October 2008; accepted 28 February 2009; published online 16 April 2009)

Escort mean values (or q-moments) constitute useful theoretical tools for describing basic features of some probability densities such as those which asymptotically decay like *power laws*. They naturally appear in the study of many complex dynamical systems, particularly those obeying nonextensive statistical mechanics, a current generalization of the Boltzmann–Gibbs theory. They recover standard mean values (or moments) for $q=1$. Here we discuss the characterization of a (nonnegative) probability density by a suitable set of all its escort mean values together with the set of all associated normalizing quantities, provided that all of them converge. This opens the door to a natural extension of the well-known characterization, for the $q=1$ instance, of a distribution in terms of the standard moments, provided that *all* of them have *finite* values. This question would be specially relevant in connection with probability densities having *divergent* values for all nonvanishing standard moments higher than a given one (e.g., probability densities asymptotically decaying as power laws), for which the standard approach is not applicable. The Cauchy–Lorentz distribution, whose second and higher even order moments diverge, constitutes a simple illustration of the interest of this investigation. In this context, we also address some mathematical subtleties with the aim of clarifying some aspects of an interesting nonlinear generalization of the Fourier transform, namely, the so-called q-Fourier transform. *© 2009 American Institute of Physics.* [DOI: 10.1063/1.3104063]

I. INTRODUCTION

Complex many-body systems with long-range interactions usually admit metastable states of long (but finite) life that eventually decay to a Boltzmann–Gibbs–like state of thermodynamical equilibrium. The life of these metastable states becomes longer as the size of the system increases. Various properties suggest that these metastable states (see Refs. 1 and 2 and references therein) may be obtained from a variational principle akin to the maximum entropy principle associated with standard Boltzmann–Gibbs thermodynamical equilibrium. Along these lines, the following entropy has been introduced:[3–5]

$$S_q[f] = \frac{1}{q-1}\left(1 - \int [f(\mathbf{x})]^q d\Omega\right), \tag{1}$$

a)Author to whom correspondence should be addressed. Electronic mail: tsallis@cbpf.br.

where $f(\mathbf{x})$ stands for a normalized probability density and $\mathbf{x}$ and $d\Omega$ denoting, respectively, a generic point and the volume element in the corresponding phase space. The parameter q determines the degree of nonadditivity exhibited by the entropic form (1). The q-thermostatistical formalism based on the entropic measure S_q has attracted considerable theoretical interest in recent years and has led to various experimental verifications of its predictions in real physical systems: see Ref. 6 for cold atoms in optical lattices,[7] for dusty plasma,[8] for the motion of *Hydra viridissima*,[9] for defect turbulence, among others. Details can be seen in available reviews[10–13] and references therein. Moreover, the q-thermostatistical formalism has proven to be a powerful theoretical tool for the treatment of a variegated family of problems in physics and other fields, ranging from the analysis of turbulence[14–17] and nonlinear diffusion processes[18–24] to the study of economic systems.[25] As mentioned above, there is an increasing body of evidence suggesting that the probability distributions maximizing S_q provide appropriate descriptions of metastable states in systems with long-range interactions.

In the limit case $q \to 1$ the entropic form (1) becomes additive and the standard Boltzmann–Gibbs–Shannon (BGS) entropy,

$$S_{\mathrm{BGS}} = S_1 = -\int f(\mathbf{x}) \ln f(\mathbf{x}) d\Omega, \tag{2}$$

is recovered. The nonadditive character of S_q is summarized in the relation

$$S_q[f^{(A+B)}] = S_q[f^{(A)}] + S_q[f^{(B)}] + (1-q) S_q[f^{(A)}] S_q[f^{(B)}], \tag{3}$$

where $f^{(A+B)}(\mathbf{x_A},\mathbf{x_B}) = f^{(A)}(\mathbf{x_A}) f^{(B)}(\mathbf{x_B})$ is the joint probability density of a composite system $A+B$ whose subsystems A and B are statistically independent and described, respectively, by the individual probability densities $f^{(A)}$ and $f^{(B)}$. The third term in the right hand side of (3) corresponds to the nonadditive behavior of S_q. When $q=1$ this term vanishes and (3) reduces to the well-known additivity relation verified by the BGS logarithmic entropy.

The probability distributions obtained maximizing the measure S_q under appropriate constraints constitute the main ingredient in the application of the q-formalism to the study of specific systems. There are several theoretical reasons suggesting that the correct constraints to use when implementing the S_q maximum entropy have to be written under the form of escort mean values (or q-mean values),

$$\langle A \rangle_q = \frac{\int A(\mathbf{x})[f(\mathbf{x})]^q d\Omega}{\int [f(\mathbf{x})]^q d\Omega}. \tag{4}$$

In particular, the quantities $A(\mathbf{x})$ whose mean values appear as natural constraints in many applications of the q-thermostatistical formalism usually have divergent linear averages $\langle A \rangle_1$. On the contrary, the quantities $A(\mathbf{x})$ provide convergent constraints if appropriate escort mean values are considered (more on this later). It is also worth mentioning that the entropy S_q, the escort constraints, and the associated Lagrange multipliers comply with a set of relations that have the same form as the celebrated Jaynes relations[26,27] connecting the entropy, the mean values, and the Lagrange multipliers appearing in the usual version of the maximum entropy principle.[28] In the particular case of Gibbs' canonical distribution (and other maximum entropy distributions appearing in equilibrium statistical mechanics), the alluded relations reduce to the well-known thermodynamic ones involving the system's entropy, the energy, temperature, and other relevant thermodynamical variables.[28]

It is a well-known mathematical fact that a probability density $f(x)$ (for simplicity's sake, we are going to consider only one dimensional situations) may be characterized by the set of mean values $\langle x^n \rangle = \int x^n f(x) dx$ $(n=1,2,3\ldots)$, whenever they are all finite and satisfy some restrictions.[40] A usual way to see this is by recourse to the Fourier transform (FT) of $f(x)$: the moment $\langle x^n \rangle$ is given by the nth derivative of the FT $F(\xi)$ of $f(x)$ (evaluated at $\xi=0$). Due to the important role played by escort mean values in the q-statistical formalism, and in many of its applications, it is of considerable interest to explore possible extensions of the above characterization of probability densities to scenarios involving densities which asymptotically decay as *power laws*. The aim of the present note is to address this problem. We shall use the q-generalization of the FT of $f(x)$ and discuss the uniqueness of its inverse and the intimately connected problem of whether a probability density could, in general, be completely determined by an appropriate set of escort mean values, whenever these are *all* finite. The latter condition is considerably less restrictive than demanding that *all* the standard mean values be finite. It is thus at this point that we open the door in the sense of generalizing the usual theorems (recovered as the $q=1$ particular case of the present study) related to the moment problem.

II. ESCORT *q*-AVERAGES AND THE CHARACTERIZATION OF PROBABILITY DENSITIES

Let $f(x)$ be a properly normalized probability density defined on the (one dimensional) variable x,

$$\int_{-\infty}^{+\infty} f(x)dx = 1. \tag{5}$$

The *un-normalized* q-moments of $f(x)$ are defined as

$$\int_{-\infty}^{+\infty} x^n[f(x)]^q dx. \tag{6}$$

On the other hand, the *normalized* q-averages (also known as escort mean values) of a given quantity $A(x)$ are

$$\langle A(x)\rangle_q = \int_{-\infty}^{+\infty} A(x)f_q(x)dx, \tag{7}$$

where $f_q(x)$ stands for the *escort probability density*,[29,30] defined as

$$f_q(x) = \frac{[f(x)]^q}{\nu_q[f]}, \tag{8}$$

where

$$\nu_q[f] = \int_{-\infty}^{+\infty} [f(x)]^q dx. \tag{9}$$

Our main instrument in order to elucidate if (and how) a probability density can be fully determined by a set of escort mean values is the q-FT. The q-FT of a normalized (non-negative) probability density $f(x)$ is defined as[31]

$$F_q[f](\xi) = \int_{-\infty}^{+\infty} dx e_q(i\xi x[f(x)]^{q-1})f(x) \quad (q \geq 1). \tag{10}$$

We recall that for real x,

$$e_q(x) \equiv e_q^x \equiv [1+(1-q)x]_+^{1/(1-q)} \quad (e_1^x = e^x), \tag{11}$$

where $[z]_+=z$ if $z\geq 0$ and vanishes if $z<0$. Noticing that an imaginary argument is needed in the q-FT, we write the latter as

$$F_q[f](\xi) = \int_{-\infty}^{+\infty} dx[1-(q-1)i\xi x[f(x)]^{q-1}]^{1/(1-q)}f(x) \quad (q \geq 1). \tag{12}$$

By taking the principal value of $[1-(q-1)i\xi x[f(x)]^{q-1}]^{1/(1-q)}$, Eq. (12) can also be recast as[31]

$$F_q[f](\xi) = \int_{-\infty}^{+\infty} dx(1+(q-1)^2\rho^2)^{1/(2(1-q))} \times \exp\left(\frac{i \arctan[(q-1)\rho]}{q-1}\right)f(x) \quad (q \geq 1), \tag{13}$$

with $\rho \equiv \xi x[f(x)]^{q-1}$. It can be verified that the derivatives of the q-FT $F_q[f](\xi)$ are closely related to an appropriate set of un-normalized q-moments of the original probability density. Indeed, the first few low-order derivatives (including the zeroth order) are given by

$$F_q[f](\xi=0) = 1, \tag{14}$$

$$\left[\frac{dF_q[f](\xi)}{d\xi}\right]_{\xi=0} = i\int_{-\infty}^{+\infty} dxx[f(x)]^q, \tag{15}$$

$$\left[\frac{d^2F_q[f](\xi)}{d\xi^2}\right]_{\xi=0} = -q\int_{-\infty}^{+\infty} dxx^2[f(x)]^{2q-1}, \tag{16}$$

and

$$\left[\frac{d^3F_q[f](\xi)}{d\xi^3}\right]_{\xi=0} = -iq(2q-1)\int_{-\infty}^{+\infty} dxx^3[f(x)]^{3q-2}. \tag{17}$$

The general n-derivative is

$$\left[\frac{d^{(n)}F_q[f](\xi)}{d\xi^n}\right]_{\xi=0} = i^n\left\{\prod_{m=0}^{n-1}[1+m(q-1)]\right\}\int_{-\infty}^{+\infty} dxx^n[f(x)]^{1+n(q-1)} \quad (n=1,2,3,\ldots). \tag{18}$$

Recalling (6), this last relation can be recast in terms of normalized q-mean moments $\langle x^n\rangle_q$,

$$\frac{1}{\nu_{q_n}}\left[\frac{d^{(n)}F_q[f](\xi)}{d\xi^n}\right]_{\xi=0} = i^n\left\{\prod_{m=0}^{n-1}[1+m(q-1)]\right\}\langle x^n\rangle_{q_n} \quad (n=1,2,3,\ldots), \tag{19}$$

with

$$q_n = 1+n(q-1). \tag{20}$$

Now, the derivatives (18) determine the form of the q-FT $F_q[f](\xi)$ through its Taylor expansion around $\xi=0$, i.e.,

$$F_q[f](\xi) = 1+\left[\frac{dF_q[f](\xi)}{d\xi}\right]_{\xi=0}\xi+\frac{1}{2}\left[\frac{d^2F_q[f](\xi)}{d\xi^2}\right]_{\xi=0}\xi^2+\frac{1}{3!}\left[\frac{d^3F_q[f](\xi)}{d\xi^3}\right]_{\xi=0}\xi^3+\cdots. \tag{21}$$

We shall address two related questions, namely, whether the inverse q-FT of $F_q[f](\xi)$ [that is, the probability density $f(x)$] is uniquely and completely determined[31] by $F_q[f](\xi)$ (see also Ref. 32), and whether *the set of quantities* ν_{q_n} and $\langle x^n\rangle_{q_n}$ do characterize completely the probability density $f(x)$. Section III will be devoted to these problems.

Naturally, Eq. (20) immediately leads to the following generalized escort distributions:

$$f_{q_n}(x) = \frac{[f(x)]^{1+n(q-1)}}{\nu_{q_n}[f]} \quad (n = 0,1,2,\ldots), \tag{22}$$

where

$$\nu_{q_n}[f] = \int_{-\infty}^{+\infty} [f(x)]^{1+n(q-1)} dx, \tag{23}$$

of which the escort distribution (8) and (9) is but the $n=1$ member.

Notice a strong property, namely, that $\langle x^n \rangle_{q_n}$ $(n=0,1,2,\ldots)$ are simultaneously *all finite* for $q<2$ and *all divergent* for $q \geq 2$, if $f(x)$ decays like $x^{1/(q-1)}$ [which, remarkably enough, is *precisely* what occurs in q-statistics, where $f(x) \propto e_q^{-\beta x}$]. Notice also that, from Eq. (20), (i) $q=1$ implies $q_n=1, \ \forall n$, thus recovering as a particular case the standard theorem about characterization of a probability density through its infinite moments; (ii) $q_1=q, \ \forall q \geq 1$, thus recovering, as another interesting particular case, the form of constraints currently used in nonextensive statistical mechanics.[28]

We now consider the typical situation arising to complex systems such as many-body problems with long-range interactions and/or quantum entanglement, edge of chaos, free-scale networks, and others (all of them being, in fact, systems typically addressed through q-thermostatistics). Usually one has probability densities behaving asymptotically as power laws,

$$f(x) \sim |x|^{-\gamma}(|x| \to \infty; \gamma > 0). \tag{24}$$

It is easy to realize that [if $f(x)$ is defined on an unbounded x-interval] the standard linear moments $\langle x^n \rangle$ will not be convergent for arbitrary values of n. Consequently, the standard way of characterizing the probability density via its linear moments is not feasible. On the other hand, let us see what happens with the set of escort mean values appearing in Eqs. (18) and (19). The normalizability of $f(x)$ and the convergence of the integrals defining the quantities ν_{q_n} and the unnormalized q_n-moments require, respectively, that the following relations hold:

$$1 - \gamma < 0,$$

$$1 - \gamma q_n = 1 - \gamma - n\gamma(q-1) < 0,$$

$$1 + n - \gamma q_n = 1 - \gamma + n[1 - \gamma(q-1)] < 0. \tag{25}$$

The above relations are verified provided that γ and q comply with

$$\gamma > 1 \tag{26}$$

and

$$q \geq 1 + \frac{1}{\gamma}. \tag{27}$$

Equation (26) can be assumed to hold because it is just the condition required for the powerlike density $f(x)$ to be normalizable. A physically interesting class of normalizable, powerlike probability densities $f(x) \sim |x|^{-\gamma}$ (like the q-Gaussians[32]) can be, if some suitable conditions are satisfied, characterized by an appropriate set of convergent escort mean values $\langle x^n \rangle_{q_n}$, as prescribed by Eqs. (18)–(20), provided that q verifies the inequality (27). We shall from now on use the most stringent value of q, namely,

$$q = 1 + \frac{1}{\gamma}, \tag{28}$$

which, as already mentioned, is consistent with q-statistics.

The above considerations can be nicely illustrated in the case of an important family of probability distributions appearing in many applications of the q-thermostatistical theory (see, for instance, Refs. 11 and 18–20 and references therein), namely, the Q-Gaussians (to avoid confusion, we adopt here the notation Q-Gaussians, instead of q-Gaussians as usually done in the literature),

$$G_Q(\beta,x) = \frac{\sqrt{\beta}}{C_Q} e_Q^{-\beta x^2}, \tag{29}$$

which are defined in terms of the Q-exponential function, which, as indicated previously, satisfies $e_Q^x \equiv [1+(1-Q)x]_+^{1/(1-Q)}$.

In Eq. (29), β is a positive parameter whose inverse $(1/\beta)$ characterizes the "width" of the Q-Gaussian and C_Q is an appropriate normalization constant. The Q-Gaussians constitute simple but important examples of maximum q-entropy (q-maxent, for short) distributions. The probability density $G_Q(\beta,x)$ maximizes the entropy S_Q under the constraints imposed by normalization and the escort mean value $\langle x^2\rangle_Q$. The parameter β is related to the Lagrange multiplier associated with the $\langle x^2\rangle_Q$ constraint. The Q-Gaussian may be regarded as a paradigmatic example of a q-maxent probability distribution. The probability density $G_Q(\beta,x)$ reduces, of course, to a standard Gaussian distribution in the limit $Q\to 1$ and recovers the Cauchy–Lorentz distribution $G_2(\beta,x)\propto 1/(1+\beta x^2)$ for $Q=2$. The distributions (29) are normalizable for $Q<3$ (the support is bounded for $Q<1$ and infinite for $1\le Q<3$). Their second moment is *finite* for $Q<5/3$ and *diverges* for $5/3\le Q<3$. But, their second Q-moment is *finite* for $Q<3$, hence *both the norm and the second* Q-moment are mathematically well defined up to the same value of Q.

Now, for Q-Gaussians we have, using Eq. (29), $G_Q(\beta,x)\propto 1/|x|^{2/(Q-1)}$ $(|x|\to\infty)$, hence $\gamma=2/(Q-1)$, with $Q>1$. Consequently, for normalizable Q-Gaussians (i.e., $Q<3$) the representation (18) and (19) can always be implemented. Since, using Eq. (28), $\gamma=1/(q-1)$, we have

$$q - 1 = \frac{Q-1}{2}, \tag{30}$$

hence, using Eq. (20), $q_n=1+n(Q-1)/2$, and therefore $q_2=Q$. This outcome precisely coincides with the well-known recipe for Q-Gaussians whenever obtained from the optimization of S_Q with fixed and finite $\langle x^2\rangle_Q$! Consistently, we verify from Eq. (30) that the well-known upper bound $Q=3$ for Q-Gaussians coincides with the upper bound $q=2$ for the present theory (and q-statistics).

III. A MATHEMATICAL ANALYSIS

A. On the nonuniqueness of the inverse to the *q*-FT

Is the inverse of $F_q[f](\xi)$, that is, the probability density $f(x)$, uniquely and completely determined?[31,32] We shall treat here the issue of the uniqueness versus nonuniqueness of the inverse to the q-FT. The difficulties induced by the nonlinearity in f in (12) will prevent us from carrying out a general analysis. Our strategy will be based on reducing the study of that nonlinearity, in general, to that in a "small" set of functions which contains the function $f(x)$ (the one determining $F_q[f]$). We shall refer to the latter study as local uniqueness versus nonuniqueness of the inverse to the q-FT. Such a strategy will allow for not necessarily small $(q-1)$ and lead to a linear problem in that "small" set of functions, still keeping certain "memory" of the basic nonlinearity. Such a "memory," in turn, will play a crucial role, as we shall see.

Our framework will make use, at a certain stage, of analytic functions. In order to motivate the latter technique, we shall comment briefly about the classical inverse moment problem ($q=1$ case) in item (1) below. We shall turn to the $q>1$ case [say, to $F_q[f](\xi)$] in item (2).

(1) Let $q=1$. We shall study briefly whether the set $\{\langle x^n\rangle_1\}$ corresponds to a unique normalized probability density $f_1(x)$ (the classical inverse moment problem).[33–36] Suppose that two square-integrable functions $f_1(x)$ and $f_2(x)$ $(-\infty<x<+\infty)$ have the same moments $\{\langle x^n\rangle_1\}$, all of which are finite. We also assume that the series $\Sigma_0^{+\infty}(n!)^{-1}(i\xi)^n\langle x^n\rangle_1$ converges (and that $\Sigma_0^{+\infty}$ and $\int_{-\infty}^{+\infty}dx$ can be interchanged) in a suitable range of ξ values (*). Then, $\epsilon(x)=f_2(x)-f_1(x)$ fulfills

$$\int_{-\infty}^{+\infty} dx x^n \epsilon(x) = 0 \quad (n=0,1,2,3,\ldots). \tag{31}$$

By subtracting the $q=1$ counterparts of Eq. (21) for both $f_1(x)$ and $f_2(x)$, it follows that

$$\int_{-\infty}^{+\infty} dx \exp i\xi x \epsilon(x) = 0. \tag{32}$$

Since $\epsilon(x)$ is in the class of square-integrable functions, Eq. (32) implies that $\epsilon(x)=0$. Then, no other normalized density $f_2(x)$ exists in the vicinity of $f_1(x)$, so that they both could have the same moments.

Let us now replace the above condition (*) by: the series $\Sigma_0^{+\infty}z^{-n}\langle x^n\rangle_1$ converges (and $\Sigma_0^{+\infty}$ and $\int_{-\infty}^{+\infty}dx$ can be interchanged) in a suitable range of z values (**). By summing a geometric series, one has for both $f_1(x)$ and $f_2(x)$,

$$\sum_0^{+\infty} \frac{\langle x^n\rangle_1}{z^n} = z\left[\int_{-\infty}^{+\infty} dx \frac{f_1(x)}{z-x}\right] = z\left[\int_{-\infty}^{+\infty} dx \frac{f_2(x)}{z-x}\right]. \tag{33}$$

Then,

$$0 = z\left[\int_{-\infty}^{+\infty} dx \frac{\epsilon(x)}{z-x}\right]. \tag{34}$$

The structure of the right-hand-side of Eq. (34) suggests that it can be extended to an analytic function in the complex z-plane, except for a discontinuity across that part of the real z axis in which $\epsilon\neq 0$. Such an analytic function has to vanish identically throughout the whole complex z-plane by virtue of the uniqueness of analytic continuation. Then, its discontinuity has to vanish as well, so that $\epsilon(x)=0$ for any x: uniqueness of the classical inverse moment problem under the assumed conditions.

At this point, we shall remind an example of nonuniqueness of the classical inverse moment problem, given by Stieltjes (quoted by Chihara[35]),

$$\frac{1}{\pi^{1/2}\exp(1/4)}\int_0^{+\infty} dx \exp(-(\ln x)^2)x^n[1+C\sin(2\pi\ln x)] = \exp\left(\frac{(n+1)^2-1}{4}\right) \equiv \langle x^n\rangle_1, \tag{35}$$

which holds for any real constant $|C|<1$. One could say that $f_1(x)$ corresponds to $C=0$ and $f_2(x)$ to $C\neq 0$. Equation (35) means that the probability distribution inside the integral gives the same classical moments for any C! Notice that the factor $\exp(-1/4)$ in Eq. (35) is aimed to ensure (at least, formally) the normalization to $+1$ of the probability distribution. Having in mind (32), we shall recast (35) into the ($q=1$) FT framework. Thus, we can write formally

$$\frac{1}{\pi^{1/2}\exp(1/4)}\int_0^{+\infty} dx\,\exp(-(\ln x)^2)\exp i\xi x[1+C\sin(2\pi\ln x)]=\sum_0^{+\infty}\frac{(i\xi)^n}{n!}\exp\left(\frac{(n+1)^2-1}{4}\right). \tag{36}$$

This would seem to imply that the whole family of functions inside the integral in (36), as the real parameter C varies (with $|C|<1$), would have the same and unique FT! Such a conclusion is invalid because the series in the right-hand side of (36) diverges, precisely due to the growth of $\exp(((n+1)^2-1)/4)$ with n. Then, in this case the condition that $\Sigma_0^{+\infty}(n!)^{-1}(i\xi)^n\langle x^n\rangle_1$ converges is not fulfilled. Similarly, the formal counterpart of Eq. (33) for the Stieltjes counterexample is

$$\frac{z}{\pi^{1/2}\exp(1/4)}\int_0^{+\infty} dx\,\exp(-(\ln x)^2)\frac{1}{z-x}[1+C\sin(2\pi\ln x)]=\sum_0^{+\infty}\frac{1}{z^n}\exp\left(\frac{(n+1)^2-1}{4}\right). \tag{37}$$

This would seem to imply that the whole family of functions inside the integral in (37), as the real parameter C varies (with $|C|<1$), would have the same and unique analytic continuation! Such a conclusion is again invalid because the series in the right-hand side of (37) diverges, for the same reason as the one in (36). The Stieltjes counterexample displays the crucial role of the convergence conditions for those series, namely, either (*) or (**) for the classical inverse moment problem. Thus, one should expect that some convergence conditions for various series in the analysis of the inverse of the q-FT in item (2) will have to be imposed.

Below, we shall be able to extend Eqs. (33) and (34) to the analysis of the inverse of the q-FT and, in Sec. III B, to the associated inverse moment problem. It seems apparent that a related analysis of the inverse of the q-FT for $q\neq 1$ in the $q=1$ FT framework [say, a generalization based on (32)] would meet difficulties.

(2) Next, let $q>1$. Our starting point will now be Eqs. (5), (18), and (21). $f(x)$ is supposed not to be wildly divergent at any finite x and to have some power-law decay for large $|x|$ [for instance, the one determined by (24), (26), and (27)]. All that is required if $f(x)$ is in the $L_1(R)$ class.[31] Suppose that two functions $f_1(x)$ and $f_2(x)$ with those properties have the same q-FT: $F_q[f_1](\xi)=F_q[f_2](\xi)$. Then, they have the same formal Taylor expansion, given in Eqs. (21). We assume that, for some domain of ξ-values, the series in Eq. (21) and the series

$$\sum_{n=0}^{+\infty}\frac{[(q-1)i\xi]^n}{1+n(q-1)}\left[\int_{-\infty}^{+\infty} dx x^n[f_j(x)]^{1+n(q-1)}\right] \tag{38}$$

converge for both $j=1,2$ [and that $\Sigma_0^{+\infty}$ and $\int_{-\infty}^{+\infty}dx$ can be interchanged and the related operations leading from (40) to (41) hold]. Such convergence conditions will play here a role similar to the condition (**) in item (1) for the classical inverse moment problem. A comparison of the factor $i^n\{\Pi_{m=0}^{n-1}[1+m(q-1)]\}/n!$ [present in Eq. (21)] with $[(q-1)i]^n(1+n(q-1))^{-1}$ [in Eq. (38)] for large n suggests that if Eq. (21) converges, then the convergence of Eq. (38) would not impose additional essential restrictions. Then $[d^{(n)}F_q[f_1](\xi)/d\xi^n]_{\xi=0}=[d^{(n)}F_q[f_2](\xi)/d\xi^n]_{\xi=0}$, not only for $n=1,2,\ldots$ but also for $n=0$, by virtue of Eq. (5). By factoring out $i^n\{\Pi_{m=0}^{n-1}[1+m(q-1)]\}$ in Eq. (18), we get

$$\int_{-\infty}^{+\infty} dx x^n[f_1(x)]^{1+n(q-1)}=\int_{-\infty}^{+\infty} dx x^n[f_2(x)]^{1+n(q-1)}\quad(n=0,1,2,3,\ldots). \tag{39}$$

Based on (38) and (39), one has

$$\sum_{n=0}^{+\infty}\frac{[(q-1)i\xi]^n}{1+n(q-1)}\left[\int_{-\infty}^{+\infty}dx x^n[f_1(x)]^{1+n(q-1)}\right]=\sum_{n=0}^{+\infty}\frac{[(q-1)i\xi]^n}{1+n(q-1)}\left[\int_{-\infty}^{+\infty}dx x^n[f_2(x)]^{1+n(q-1)}\right]. \tag{40}$$

For fixed $f_1(x)$, let $f_2(x)-f_1(x)=\epsilon(x)$ be small [for instance, let the norm of $\epsilon(x)$ be suitably small compared to that of $f_1(x)$ in the $L_1(R)$ class]. This will implement our search of local uniqueness versus nonuniqueness of the inverse to the q-FT in a "small" set of functions (the one formed by all such f_2's) about f_1, as announced at the beginning of this subsection. Equation (40) yields, by expanding its right-hand side into powers of $\epsilon(x)$ inside the integrals, keeping only the first order in $\epsilon(x)$ and summing a geometric series,

$$0=\sum_{n=0}^{+\infty}[(q-1)i\xi]^n\left[\int_{-\infty}^{+\infty}dx x^n[f_1(x)]^{n(q-1)}\epsilon(x)\right]=\int_{-\infty}^{+\infty}dx\frac{\epsilon(x)}{1-i\xi(q-1)xf_1(x)^{q-1}}\equiv H_1(\xi). \tag{41}$$

Notice the formal similarity between $H_1(\xi)$ in Eqs. (41) and (12), except for the crucial exponent $1/(1-q)$ in the latter. $H_1(\xi)$ does not coincide with $F_q[f](\xi)$, but it will provide a useful framework to discuss local uniqueness versus nonuniqueness of the inverse to the q-FT. If $z=[i\xi(q-1)]^{-1}$, Eq. (41) can be recast as

$$G_1(z)=H_1(\xi)=z\left[\int_{-\infty}^{+\infty}dx\frac{\epsilon(x)}{z-xf_1(x)^{q-1}}\right], \tag{42}$$

which is the $q\neq 1$ counterpart of Eq. (34). The structure of Eq. (42) suggests that $G_1(z)$, which vanishes by virtue of Eqs. (41) and (42), can be extended to an analytic function in the complex z-plane, except for a discontinuity across the real z axis. Such an analytic function has to vanish identically throughout the whole complex z-plane by virtue of the uniqueness of analytic continuation. If one could infer that $G_1(z)\equiv 0$ implies $\epsilon(x)\equiv 0$, that would indicate the local uniqueness of the inverse to the q-FT, in a "small" set of functions f_2 which contains f_1. However, this will not be the case, in general, as we shall see, due to the key structure $xf_1(x)^{q-1}$. This is the "memory" of the genuine nonlinearity of the q-FT, mentioned at the beginning of this subsection.

The following example will clarify the issue. We turn to the following class of normalizable non-negative probability densities $f_1(x)$, $-\infty<x<+\infty$, with the following properties: (1) $f_1(-x)=f_1(x)$, (2) $f_1(0)$ is finite, and (3) $f_1(x)$ decreases monotonically in $0<x<+\infty$, with $f_1(x)\to 0$ as $x\to+\infty$. This class appears to include the Cauchy–Lorentz distribution. As $f_1(x)^{q-1}$ decreases monotonically in $0<x<+\infty$, it follows that $xf_1(x)^{q-1}$ vanishes at $x=0$, increases monotonically in $0<x<x_0$, and decreases monotonically in $x_0<x<+\infty$. The value x_0 is defined so that $xf_1(x)^{q-1}$ takes on its maximum (denoted as $y_0>0$), at $x=x_0$. Then, in $0<x<+\infty$, the function $xf_1(x)^{q-1}=y$ has two inverses, namely, $x_1(y)$ and $x_2(y)$, with $0\leq y\leq y_0$ ($dx_1/dy>0$ and $dx_2/dy<0$). One has

$$G_1(z)=G_{1,+}(z)+G_{1,-}(z), \tag{43}$$

$G_{1,-}(z)$ and $G_{1,-}(z)$ are the contributions from $0<x<+\infty$ and $0>x>-\infty$, respectively. As $G_1(z)\equiv 0$ and $G_{1,+}(z)$ and $G_{1,-}(z)$ have different domains of discontinuity, it follows that $G_{1,+}(z)=G_{1,-}(z)\equiv 0$. For $\epsilon(x)$ in $0<x<+\infty$, one has

$$G_{1,+}(z)=z\left[\int_0^{x_0}dx\frac{\epsilon(x)}{z-xf_1(x)^{q-1}}+\int_{x_0}^{+\infty}dx\frac{\epsilon(x)}{z-xf_1(x)^{q-1}}\right]. \tag{44}$$

By performing the change in variables $x\to y$,

$$G_{1,+}(z) = z\left[\int_0^{y_0} dy \frac{(dx_1/dy)\epsilon(x_1(y)) + (dx_2/dy)\epsilon(x_2(y))}{z-y}\right]. \tag{45}$$

Since $G_{1,+}(z)\equiv 0$, it follows that $(dx_1/dy)\epsilon(x_1(y))+(dx_2/dy)\epsilon(x_2(y))=0$ for any $0<y<y_0$. But this does not require that $\epsilon(x_1(y))=0$ and $\epsilon(x_2(y))=0$ separately for any $0<y<y_0$, that is, there may be a cancellation between $(dx_1/dy)\epsilon(x_1(y))$ and $(dx_2/dy)\epsilon(x_2(y))$ due to the different signs of dx_1/dy and dx_2/dy. That is, $\epsilon(x)$ with $0<x<+\infty$ is not forced to vanish. $G_{1,-}(z)$, for $\epsilon(x)$ in $0>x>-\infty$, can be treated similarly and leads to the same conclusion.

Then, there is no, in general, local uniqueness of the inverse to the q-FT. On the other hand, local uniqueness of the inverse to the q-FT holds indeed for restricted classes of functions: one of such classes is that formed by q-Gaussians.

B. On the characterization of a probability density by all escort mean values $\langle x^n\rangle_{q_n}$'s together with all ν_{q_n}'s

We now investigate whether a probability density $f(x)$ can be uniquely characterized by the set of all escort mean values $\langle x^n\rangle_{q_n}$ together with the set of all associated quantities ν_{q_n}. Suppose that two probability densities $f_1(x)$ and $f_2(x)$ have the same $\langle x^n\rangle_{q_n}$ and the same $\nu_{q_n}[f_1]=\nu_{q_n}[f_2]$ [Eqs. (5) and (23)] for all $n=0,1,2,\ldots$. We continue to make the same assumptions on $f_1(x)$ and $f_2(x)$ as in item (2) of Sec. III B, so that Eqs. (39) and (40) hold. We shall add the following condition: the series

$$\sum_{n=0}^{+\infty} \frac{[(q-1)i\xi]^n}{1+n(q-1)}\left[\int_{-\infty}^{+\infty} dx[f_j(x)]^{1+n(q-1)}\right] \tag{46}$$

converge for both $j=1,2$ [and, again, $\Sigma_0^{+\infty}$ and $\int_{-\infty}^{+\infty}dx$ can be interchanged and the related operations leading from (47) to (49) hold] for some domain of ξ-values. The convergence assumptions for Eqs. (38) and (46) play a role for the actual $q>1$ moment problem similar to the convergence assumption (**) for the classical moment problem in item (1) of Sec. III B. As $\nu_{q_n}[f_1]=\nu_{q_n}[f_2]$ for $n=0,1,2,\ldots$, one has

$$\sum_{n=0}^{+\infty} \frac{[(q-1)i\xi]^n}{1+n(q-1)}\left[\int_{-\infty}^{+\infty} dx[f_1(x)]^{1+n(q-1)}\right] = \sum_{n=0}^{+\infty} \frac{[(q-1)i\xi]^n}{1+n(q-1)}\left[\int_{-\infty}^{+\infty} dx[f_2(x)]^{1+n(q-1)}\right]. \tag{47}$$

Let $f_2(x)-f_1(x)=\epsilon(x)$ is small, so that one recovers Eq. (42). Moreover, by using the same arguments as in item (2) of Sec. III B, with $xf_1(x)^{q-1}$ replaced by $f_1(x)^{q-1}$, one gets

$$\int_{-\infty}^{+\infty} dx[f_1(x)]^{n(q-1)}\epsilon(x) = 0 \quad (n=0,1,2,3,\ldots). \tag{48}$$

Moreover, Eq. (47) yields ($z=[i\xi(q-1)]^{-1}$),

$$0 = H_2(\xi) = \sum_{n=0}^{+\infty}[(q-1)i\xi]^n\left[\int_{-\infty}^{+\infty} dx[f_1(x)]^{n(q-1)}\epsilon(x)\right] = z\left[\int_{-\infty}^{+\infty} dx\frac{\epsilon(x)}{z-f_1(x)^{q-1}}\right] = G_2(z). \tag{49}$$

Both $G_1(z)$ and $G_2(z)$ can be extended to analytic functions in the complex z-plane. On the other hand, they both have to vanish identically throughout the whole complex z-plane. Then, the discontinuities of both $G_1(z)$ and $G_2(z)$ across the real z axis will provide two conditions on $\epsilon(x)$ and the question is whether they suffice to ensure $\epsilon(x)\equiv 0$.

We consider again the same class of normalizable non-negative probability densities $f_1(x)$, $-\infty<x<+\infty$ as in the example in the last part of Sec. III B, which led to Eq. (43) and to the nonuniqueness to the inverse of the q-FT. We start with $G_2(z)$, which reads ($q>1$),

$$G_2(z) = z\left[\int_0^{+\infty} dx \frac{\epsilon(x) + \epsilon(-x)}{z - f_1(x)^{q-1}}\right] \tag{50}$$

as $f_1(-x)^{q-1} = f_1(x)^{q-1}$. Since $f_1(x)^{q-1}$ is monotonic, $G_2(z)=0$ implies $\epsilon(x) = -\epsilon(-x)$, to be used in what follows. We shall now consider

$$G_1(z) = G_{1,+}(z) + G_{1,-}(z) = z\left[\int_0^{+\infty} dx \frac{\epsilon(x)}{z - xf_1(x)^{q-1}} - \int_{-\infty}^{0} dx \frac{\epsilon(-x)}{z - xf_1(x)^{q-1}}\right], \tag{51}$$

Where $G_{1,+}(z)$ and $G_{1,-}(z)$ are the first and second integrals in the right-hand side of Eq. (43), respectively. As the ranges of discontinuity of $G_{1,+}(z)$ and $G_{1,-}(z)$ are disjoint, $G_{1,+}(z) \equiv 0$ and $G_{1,-}(z) \equiv 0$ follow. By performing the same change in variables $x \to y$ which led to Eq. (45),

$$G_{1,+}(z) = z\left[\int_0^{y_0} dy \frac{(dx_1/dy)\epsilon(x_1(y)) + (dx_2/dy)\epsilon(x_2(y))}{z - y}\right]. \tag{52}$$

As $G_{1,+}(z) \equiv 0$, it follows that $(dx_1/dy)\epsilon(x_1(y)) + (dx_2/dy)\epsilon(x_2(y)) = 0$ for any $0 < y < y_0$. As there may be a cancellation between $(dx_1/dy)\epsilon(x_1(y))$ and $(dx_2/dy)\epsilon(x_2(y))$, $\epsilon(x)$ is not forced to vanish. The consideration of $G_{1,-}(z)$ leads to a similar conclusion.

Then, a probability density does not appear to be characterized uniquely by the set of all its escort mean values $\langle x^n \rangle_{q_n}$'s together with all its ν_{q_n}'s, in general. However, as we already mentioned earlier, the convenient feature of uniqueness might occur for special classes of physically relevant densities, with special constraints.

IV. CONCLUSIONS

We have argued that the q-FT, which is a crucial tool for these studies, has not a unique inverse, in general. Intimately connected to that, we have argued also that a (non-negative) probability density $f(x)$ cannot be, in general, fully characterized by the set of all escort mean values $\langle x^n \rangle_{q_n}$ together with the set of all associated quantities ν_{q_n}, which are the integrals of the q_n-powers of the density $f(x)$. However, for specific classes of inverses, depending typically on a set of generic coefficients, the use of the set $\{\nu_{q_n}\}$, together with all the q-moments, is expected to be sufficient for uniquely determining the physically appropriate inverse. It is of course required that all those escort mean values and all ν_{q_n}'s converge. Section III has focused on these mathematical subtleties.

The exponents q_n are given by $q_n = 1 + n(q-1)$. For the important case of powerlike probability densities [i.e., $f(x)$ decaying like $1/|x|^\gamma$ for $|x| \to \infty$, with $\gamma > 1$], we have determined the range of q-values [inequality (27)] for which all the alluded quantities are finite. The particular case $q=1$ recovers the usual connection (applicable only to distributions such that *all* the standard moments are *finite*). Making the choice $\gamma = 1/(q-1)$, the whole construction is mathematically admissible for $q < 2$, in full consistency with the q-exponential distribution proportional to $e_q^{-\beta x}$, naturally emerging within nonextensive statistical mechanics. In other words, although this connection implies the use of auxiliary conditions and is subject to mathematical subtleties, it is completely independent of nonextensive statistics. In fact, it enriches the current use[28,30] of escort distributions in the definition of the constraints under which the entropy S_q is to be optimized, to obtain the stationary-state distribution.

In the present contribution we have considered representations of one dimensional probability densities in terms of escort mean values of powers of the state variable x. It would be interesting to extend this approach to higher dimensional situations and to consider escort mean values associated with other functions of the state variables, such as polynomials. These extensions may be useful for the study of time dependent processes in complex systems (e.g., systems with long-range interactions) using hierarchies of evolution equations derived from the corresponding Liouville equation (see, for instance, Refs. 37 and 38). These lines of inquiry will be addressed in a future publication.

Let us finally point out that further mathematical investigations would be interesting regarding (i) the precise radius of convergence of the expansion (21) (it is nevertheless already clear that this radius is not zero, since it contains the q-Gaussian distributions[32,39]); and (ii) a more extended analysis about the precise classes of functions for which the q-FT is either invertible or noninvertible, and about the precise class of probability densities $f(x)$ which are uniquely determined by the set of all escort mean values $\langle x^n \rangle_{q_n}$'s together with the set of all ν_{q_n}'s.

ACKNOWLEDGMENTS

One of us (C.T.) has benefited from interesting related conversations with H. J. Hilhorst, S. Umarov, and E. M. F. Curado, and acknowledges partial financial support by CNPq and Faperj (Brazilian agencies). Another one (R.F.A.-E.) acknowledges the financial supports of Ministerio de Educacion y Ciencia (Project No. FPA2004-02602) and of Ministerio de Ciencia e Innovacion (Project No. FIS2008-01323), Spain. This work was partially supported (A.R.P.) by the Projects FQM-2445 and FQM-207 of the Junta de Andalucia (Spain, EU).

[1] A. Pluchino, A. Rapisarda, and C. Tsallis, Europhys. Lett. **80**, 26002 (2007).
[2] A. Pluchino, A. Rapisarda, and C. Tsallis, Physica A **387**, 3121 (2008).
[3] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[4] C. Tsallis, Phys. World **10**, 42 (1997).
[5] C. Tsallis, Braz. J. Phys. **29**, 1 (1999).
[6] P. Douglas, S. Bergamini, and F. Renzoni, Phys. Rev. Lett. **96**, 110601 (2006).
[7] B. Liu and J. Goree, Phys. Rev. Lett. **100**, 055003 (2008).
[8] A. Upadhyaya, J.-P. Rieu, J. A. Glazier, and Y. Sawada, Physica A **293**, 549 (2001).
[9] K. E. Daniels, C. Beck, and E. Bodenschatz, Physica D **193**, 208 (2004).
[10] *Nonextensive Entropy—Interdisciplinary Applications*, edited by M. Gell-Mann and C. Tsallis (Oxford University Press, Oxford, 2004).
[11] *Nonextensive Statistical Mechanics: New Trends, New perspectives*, Europhysics News, edited by J. P. Boon and C. Tsallis (European Physical Society, 2005), Vol. 36.
[12] C. Tsallis, *Entropy*, in *Encyclopedia of Complexity and Systems Science* (Springer, Berlin, 2009).
[13] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics—Approaching a Complex World* (Springer, New York, 2009).
[14] C. Beck, Phys. Rev. Lett. **87**, 180601 (2001).
[15] C. Beck, Europhys. Lett. **57**, 329 (2002).
[16] H. Touchette and C. Beck, Phys. Rev. E **71**, 016131 (2005).
[17] C. Beck, Phys. Rev. Lett. **98**, 064502 (2007).
[18] A. R. Plastino and A. Plastino, Physica A **222**, 347 (1995).
[19] C. Tsallis and D. J. Bukman, Phys. Rev. E **54**, R2197 (1996).
[20] T. D. Frank, *Nonlinear Fokker-Planck Equations* (Springer-Verlag, Berlin, 2005).
[21] E. K. Lenzi, L. C. Malacarne, R. S. Mendes, and I. T. Pedron, Physica A **319**, 245 (2003).
[22] F. D. Nobre, E. M. F. Curado, and G. Rowlands, Physica A **334**, 109 (2004).
[23] V. Schwammle, E. M. F. Curado, and F. D. Nobre, Eur. Phys. J. B **58**, 159 (2007).
[24] V. Schwammle, F. D. Nobre, and E. M. F. Curado, Phys. Rev. E **76**, 041123 (2007).
[25] L. Borland, Phys. Rev. Lett. **89**, 098701 (2002).
[26] E. T. Jaynes, in *Papers on Probability, Statistics and Statistical Physics*, edited by R. D. Rosenkrantz (Reidel, Dordrecht, 1987).
[27] E. T. Jaynes, *Probability Theory: The Logic of Science* (Cambridge University Press, Cambridge, 2005).
[28] C. Tsallis, R. S. Mendes, and A. R. Plastino, Physica A **261**, 534 (1998).
[29] C. Beck and F. Schlogl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, 1993).
[30] S. Abe, Phys. Rev. E **68**, 031101 (2003).
[31] S. Umarov, C. Tsallis, and S. Steinberg, Milan J. Math. **76**, 307 (2008).
[32] S. Umarov and C. Tsallis, Phys. Lett. A **372**, 4874 (2008).
[33] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments, Am. Math. Soc. Mathematical Surveys* (American Mathematical Society, New York, 1943), Vol. II.
[34] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis* (Hafner, New York, 1965) (translated by N. Kemmer).
[35] T. S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).
[36] A. G. Bakan, Proc. Am. Math. Soc. **130**, 3545 (2002).
[37] R. F. Alvarez-Estrada, Ann. Phys. **11**, 357 (2002); **15**, 379 (2006).
[38] R. F. Alvarez-Estrada, Eur. Phys. J. A **31**, 761 (2007).
[39] C. Tsallis and S. M. D. Queiros, AIP Conf. Proc. **965**, 8 (2007); S. M. D. Queiros and C. Tsallis, *ibid.* **965**, 21 (2007).
[40] The so-called moment problem in theory of probabilities is a mathematically quite complex one. The discussion is normally done separately for various classes of support of the probability distribution, namely, for the $[0,1]$ support (*Hausdorff* moment problem), the $[0,\infty)$ support (*Stieltjes* moment problem), and the $(-\infty,\infty)$ support (*Hamburger* moment problem). General necessary and sufficient conditions are still elusive (Refs. 33–36).